Non-Linear Diffusion Equation and Relaxation Processes in Solids

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Abstract. A non-logarithmic time dependence of the magnetization relaxation in superconductors is modeled by certain solutions of the nonlinear equation for flux diffusion, which can be derived either on the assumption of a logarithmic dependence of the pinning potential on the current density or, equivalently, a power-law current-voltage characteristic. Limiting cases of the spatiotemporal evolution of the flux density profile are identified: at one end of the parameters governing non-linearity the classical, linear processes relevant to the reversible part of the H-T diagram of high-T_c materials are found, whereas at the other end, a true critical-state behaviour emerges. Scaling relations between the sample size, magnetic field and characteristic relaxation times are established, which should characterize the magnetization relaxation process as well as the response to an AC field in susceptibility measurements.

1 Introduction

There is growing experimental evidence for a non-logarithmic time dependence of the magnetization relaxation in superconductors. The simple picture of a single-barrier thermally activated process of vortex diffusion, formulated for classical superconductors, is replaced by models invoking a collective vortex response to the magnetic field. As it will be shown below for a special case, the assumption of a current-dependent pinning potential U(j) leads to the same description of flux penetration as that of a non-linear current-voltage characteristic E(j). In both cases, the Maxwell equations take the form of a non-linear diffusion equation, which, together with the time-dependent boundary field, determines the spatio-temporal evolution of the flux density profile in a superconductor. The present work has been motivated by approximate, analytical results found recently for the case when the flux-pinning potential depends logarithmically on the current, $U=U_c\ln(j_c/j)$ [1,2]. The predictions based on these results have been verified experimentally by magnetization relaxation studies on single-crystal samples of the high-T_c superconductor $Bi_2Sr_2CaCu_2O_8$ [3]. Here, we present certain new results, corresponding to power-type j-E characteristics and obtained for an idealized, one-dimensional situation, when the external field is applied parallel to the surface of a large superconducting plate. The choice of the E(j) dependence for our studies is dictated by the increasing number of experimental data supporting the validity of the power-law j-E function for high-T_c superconductors and by the equivalence of the resulting model to the one in ref [1]. This model and hence our results are relevant to a variety of physical phenomena described by the nonlinear diffusion equation discussed here. In particular, they apply to a superconductor at the Kosterlitz-Thouless transition or near the irreversibility line.

2 Nonlinear Diffusion Equation

If the conductivity σ of a substance depends on the electric field as

$$j = \sigma(E) \cdot E = \sigma_0 \cdot E_0 \cdot (E/E_0)^{1-\alpha}\,, \tag{1}$$

where $0 \leq \alpha < 1$, then, using Maxwell's equations, $\bigtriangledown x B = 4\pi/c \cdot j$ and $c \bigtriangledown x E = -\partial B/\partial t$, one derives the nonlinear
diffusion equation describing the penetration of fields into a slab of thickness 2D, made of that substance, lying in the yz plane:

$$\frac{\partial \beta}{\partial \bar{t}} = \frac{\partial}{\partial \bar{x}} \left(\frac{\partial \beta}{\partial \bar{x}} \cdot |\frac{\partial \beta}{\partial \bar{x}}|^{\frac{\alpha}{1-\alpha}} \right)\,, \qquad \bar{t} = \frac{t}{\tau_0}, \quad \bar{x} = \frac{x}{x_0}, \tag{2}$$

where $\beta = B/E_0$, while x_0 and τ_0 are given in Table I.
Equation (2) is obtained also if it is assumed that the vortex pinning potential U is given by $U = U_c \cdot \ln(j_c/j)$ [1]. This can be verified by taking into account that the vortex motion induces an electric field, given by cE=B x v, and using the expression $v = v_0 \cdot (j/j_c) \cdot \exp(-U(j)/k_B T)$ for the vortex velocity v [1]. One then obtains a relation between the electric field induced by the flux motion and the supercurrent: $cE = v_0 B (j/j_c)^{\kappa+1}$, where $\kappa = U_c/k_B T$. This result can be written in the form (1) with the following assignments: $E_0 = v_0 B/c$, $\sigma_0 = j_c/E_0$ and $\kappa = \alpha/(1-\alpha)$. From this identification it is clear that the parameters E_0 and σ_0 depend on the magnetic flux density B. However, this dependence can be neglected if a small perturbation field ΔB is superimposed on a large bias field B_0, $\Delta B(x,t) \ll B_0$ and $B(x,t) = B_0 + \Delta B(x,t)$. In that case the flux evolution is again described by eq. (2) [1], with the length-, time- and field scale factors as is given in the second column of Table I. The assumed form for the vortex velocity results from the requirement that at large fields, in the reversible region of the H-T phase diagram, when $U_c/k_B T \ll 1$, the usual flux-flow equations can be recovered. We should assign $v_0/c \cdot B_{c2}\sigma_n$ to j_c, to obtain the flux-flow expression for the electric conductivity, $\sigma_0 = \sigma_n \cdot B_{c2}/B$.

TABLE I
DIMENSIONAL UNITS IN EQUATION 2

Nonlinear j-E dependence	Logarithmic pinning potential
$x_0=c/(4\pi\sigma_0)$	$x_0=cB_0/(4\pi j_c)$
$\tau_0=1/(4\pi\sigma_0)$	$\tau_0=x_0/\nu_0$
$\beta=B/E_0$, $\beta_0=E_0$	$\beta=(B-B_0)/B_0$, $\beta_0=B_0$

3 Approximate Solutions

3.1 The Propagating Flux Profile

The simplest solution corresponding to (2) has the form of a product of powers of x and t and it can be written in the form:

$$B(x,t) = b_0 \cdot \xi^{(2-\alpha)/\alpha}, \tag{3}$$

where $b_0=(2/(1-\alpha))\cdot [\alpha(1-\alpha)/(2(2-\alpha))]^{1/\alpha}\cdot \beta_0$, $\xi=(x-d)/x_0 \cdot (t/\tau_0)^{-(1-\alpha)/(2-\alpha)}$, and β_0 is defined in Table I. This is, however, a result which is only valid with a time-dependent field at the surface; this field can be obtained after the surface coordinate, D, is substituted for x in (3). A step-like time dependence, represented by the Heaviside function, $B(D,t)=H_0\cdot \Theta(t)$ comes closest to the experimental situation. The function (3) can be made to match this field at the surface by the change of variables $\xi\rightarrow\xi+(H_0/b_0)^{\alpha/(2-\alpha)}$, setting $d=\pm D$ and chosing the upper and lower sign for $x>0$ and $x<0$, respectively. However, the resulting functions are not exact solutions of (2). This can be easily seen from the form [1]

$$\xi = \frac{(|x|-x_f)/x_0}{(t/\tau_0)^{(1-\alpha)/(2-\alpha)}}, \tag{4}$$

of the new variable, where $x_f=D-x_0[(H_0/b_0)^{\alpha}(t/\tau_0)^{1-\alpha}]^{1/(2-\alpha)}$ is a function of time. The function

$$\begin{aligned} B(x,t) = \;& b_0\xi^{(2-\alpha)/\alpha} \qquad x_f(t) \le |x| \le D \\ =\;& 0 \qquad\qquad\quad |x| < x_f(t), \end{aligned} \tag{5}$$

describes flux profiles propagating from the sample surface at x=±D towards the centre at x=0, with the flux fronts B=0 at |x|=x_f(t), but it violates (2) through the time depen-dence of x_f. These results are not applicable to times t>t*, where the crossover time t*, when the flux profile reaches the sample centre, is determined from the condition $x_f(t^*)=0$ and is given by $t^*/\tau_0=(D/x_0)^{(2-\alpha)/(1-\alpha)}\cdot(b_0/H_0)^{\alpha/(1-\alpha)}$. Then the boundary condition ∂B/∂x=0 at x=x_f is to be replaced by ∂B/∂x=0 at x=0, whereas the requirement B=H_0 at x=D remains unchanged. Again, the general solution (3) has to be modified in such a way that the resulting function does not exactly satisfy (2) any more. Setting again d=±D and adding the constant H_0, one gets

$$B(x,t) = H_0 \cdot \left[1 - (1 - |x|/D)^{(2-\alpha)/\alpha} \cdot (t/t^*)^{-(1-\alpha)/\alpha}\right] , \tag{6}$$

which satisfies the boundary condition at x=D, but fails to do so at x=0. Another approximate solution,

$$B(x,t) = H_0 \cdot \left[1 - \left(1 - (|x|/D)^{(2-\alpha)/\alpha}\right) \cdot (t/t^*)^{-(1-\alpha)/\alpha}\right] , \tag{7}$$

has been suggested [1] and is readily seen to match the function (5) exactly at t=t*.

3.2 Magnetization

The magnetization is found by calculating the volume average of the flux density. For the short-time relaxation, t<t*, after integration of function (5), we have:

$$4\pi M_s(t) = \left[\left(1 - \frac{x_f(t)}{D}\right)\frac{\alpha}{2} - 1\right] \cdot H_0 = \left[\frac{\alpha}{2} \cdot \left(\frac{t}{t^*}\right)^{\frac{1-\alpha}{2-\alpha}} - 1\right] \cdot H_0 , \tag{8}$$

For the long-time relaxation, integration of (7) gives the magnetization M_L(t):

$$4\pi M_L(t) = -H_0 \cdot (1 - \alpha/2) \cdot (t/t^*)^{-(1-\alpha)/\alpha} . \tag{9}$$

One can verify, using the result on t*, that the long-time magnetization is independent of H_0. While for t<t* the relaxation curve -?- M(t) -?- has a negative curvature, d^2 -?- M(t) -?- $/dt^2<0$, for t>t* this curvature becomes positive, as observed in experiments on superconductors [3] and also on other magnetic systems.

The above results are approximate only. In particular, it follows from (8) and

(9) that the relaxation rate dM/dt has a jump at t=t*, which only disappears in the limit $\alpha \nearrow 1$: (dM_s/dt)/(dM_L/dt)=(α/(2-α))^2. That this is an unphysical result can be seen if one uses (2) to evaluate the time derivative of the magnetization: (d/dt)$\int$(B(x,t)-H_0)dx/(2D) = $x_0{}^{\kappa+2}/(D\tau_0\beta_0{}^{\kappa})$ $\cdot(\partial B/\partial x)^{\kappa+1}$ - ?- $^{D}{}_{-D}$. Thus, for any solution of (2) a jump in dM/dt implies a discontinuous change of the current density j=-(c/4π)$\cdot\partial$B/∂x at the surfaces, which cannot be expected to occur when two flux-fronts meet at x=0. In fact, it is easily verified that the current density implied by (5) and (7) is continuous everywhere, so that the discontinuity in dM/dt is a measure of the inadequacy of these functions as solutions of (2).

4 An Exact Result

4.1 Solution at t<t* by Boltzmann's transformation

In the present section we shall show that an exact solution corresponding to (2) can be found for t<t*, using the transformation introduced by Boltzmann [4]. This transformation is based on the Ansatz β=$\beta(\xi)$, where ξ=ξ(x,t) and reduces the partial differential equation (2) to an ordinary differential equation. Substituting the Ansatz into equation (2) leads to

$$\frac{\partial \xi}{\partial t} \cdot \frac{d\beta}{d\xi} = (\kappa+1) \cdot \left(\frac{d\beta}{d\xi} \cdot \frac{\partial \xi}{\partial x} \right)^{\kappa} \cdot \left(\frac{d^2\beta}{d\xi^2} \left(\frac{\partial \xi}{\partial x} \right)^2 + \frac{d\beta}{d\xi} \frac{\partial^2 \xi}{\partial x^2} \right) , \tag{10}$$

with κ=α/(1-α) or κ=U_c/k_BT. Equation (4) provides a convenient choice for ξ, as $\partial^2\xi/\partial x^2$=0 in this case. An equivalent form of (4) is

$$\xi = a \cdot \frac{-(1-x/d) + (t/t^*)^{1/(\kappa+2)}}{(t/t^*)^{1/(\kappa+2)}} , \tag{11}$$

where the parameters a, d and t* are still to be determined. Equations (10) and (11) will describe a propagating flux profile, if the boundary condition $\beta(0)$=0 is imposed. As before, we find that d=D ensures a constant field at the surface. The flux front is then seen to move according to x_f/D=1-(t/t*)$^{1/(\kappa+2)}$. Furthermore, substitution of (11) into (10) yields the simplest result, viz.

$$-\xi + a = \frac{d}{d\xi} \left(\frac{d\beta}{d\xi} \right)^{\kappa} , \tag{12}$$

if t* and a are related as

$$\frac{t^*}{\tau_0} = \frac{\kappa}{\kappa+1} \cdot \frac{1}{(\kappa+2)\, a^{\kappa+2}} \left(\frac{D}{x_0}\right)^{\kappa+2} . \tag{13}$$

The first integration of (12) gives:

$$d\beta/dy = -(1/2)^{1/\kappa+1}\, a^{2/\kappa+1}\, y^{-1/2}\, (1-y)^{1/\kappa} , \tag{14}$$

where we defined: $y=(1-\xi/a)^2$. We assumed also $d\beta/d\xi=0$ at $\xi=0$, in order to ensure a vanishing current density at the flux front. To integrate (14), we could expand the right-hand side in a power series of y and then perform an integration of this series. However, the right-hand side of (14) may be expressed with the help of a hypergeometric function [5], $(1-y)^{1/\kappa}=F(-1/\kappa,b;b;y)$, where b is an arbitrary number. We can use then a proper integration formula for hypergeometric functions [5], with the following result: $\beta(y)=-a^{2/\kappa+1}(1/2)^{1/\kappa}\cdot y^{1/2}\cdot F(-1/\kappa,1/2;3/2;y)+C$, where C is an integration constant. Imposing the boundary condition at $\xi=0$, $\beta(y=1)=0$, and using the relation between F(a,b;c;1) and Euler's gamma function Γ [5], we can write:

$$\beta(y) = \left(1 - \frac{\Gamma\,(1/\kappa+3/2)}{\Gamma\,(1/\kappa+1)\,\Gamma\,(3/2)} \cdot y^{1/2}\, F(-1/\kappa,\, 1/2;\, 3/2;\, y)\right) \cdot \beta_d\, , \tag{15}$$

where $\beta_d=a^{2/\kappa+1}(1/2)^{1/\kappa}\cdot\Gamma(1/\kappa+1)\Gamma(3/2)/\Gamma(1/\kappa+3/2)$. This equation gives the relation between the parameter a, being not specified up to now, and the normalized external field $\beta_d=H_0/\beta_0$. We use it to rewrite (13) to the form:

$$\frac{t^*}{\tau_0} = \frac{\kappa}{2(\kappa+1)}\frac{1}{(\kappa+2)}\left[\frac{\Gamma(1/\kappa+1)\,\Gamma(3/2)}{\Gamma(1/\kappa+3/2)}\,\frac{\beta_0}{H_0}\right]^{\kappa} \cdot \left(\frac{D}{x_0}\right)^{\kappa+2} . \tag{16}$$

Equation (15) constitutes an exact solution of the nonlinear flux-diffusion equation for the situation, when the external field is changed abruptly at the surface at x=D. The solution describes the propagation of the flux profile into the sample and is valid for the half-filled space as well as for a slab of thickness 2D at $t<t^*$.

We observe that t^* approaches zero in the limit of a linear diffusion equation, when $\kappa\ll1$ is substituted into (16). That indeed corresponds to the expected behavior, since in this limit the profile of flux density propagates immediately into the whole volume of the medium, i.e. a critical position x_f, beyond which $\beta=0$, cannot be found in the linear case [4,6]. It is easier to show the proper extrapolation of (14) to the linear limit, rather than that of (15). One obtains for $\kappa\ll1$ that (14) can be written in the form $d\beta/d\psi=-2\pi^{-1/2}\cdot\exp(-\psi^2)$, with

ψ=(1-x/D)/(2(t/τ_0)$^{1/2}$)·(x_0/D), which is the result derived by Boltzmann [4] for the linear case.
An important difference between the exact (15) and the approximate (5) results is that the current at the edge of the flux profile x=x_f(t) is equal to zero in (15) (although there is a rapid, but monotonic increase from j=0 at x≤x_f to a value nearly independent of x at x>x_f), while an unphysical, discontinuous jump in j(x_f) is obtained from (5).
Another interesting choice for the variable ξ(x,t), which satisfies the condition $\partial^2\xi/\partial x^2$=0 and gives an easily solvable differential equation for $\beta(\xi)$ is ξ=ax+vt. Then Bean's critical-state situation is recovered in the limit of $\alpha\approx1$, which is appropriate for the case of strong flux pinning: an external field increasing linearly with time at the surface leads to a linear flux profile β(x) spreading out uniformly toward the centre. We observe also that in this limit results for β(x,t) are not very sensitive to the way, in which the field is turned on, but depend on the instantaneous boundary field value only.

4.2 Magnetization

In order to calculate the magnetization, we have to compute the following integral:

$$\frac{1}{D}\int_{x_f}^{D}\beta(x,t)dx=\frac{1}{2}\left(\frac{t}{t^*}\right)^{1/(\kappa+2)}\cdot\int_0^1\beta(y)\,y^{-1/2}\,dy\,, \qquad (17)$$

with β(y) given by (15). Finally, we derive the magnetization

$$4\pi M\;=-H_0\cdot\left(1-\left(\frac{t}{t^*}\right)^{1/(\kappa+2)}\cdot\frac{\Gamma\;(1/\kappa+3/2)}{\Gamma\;(1/\kappa+2)\;\cdot\;\Gamma\;(1/2)}\right)\,. \qquad (18)$$

Although t^* approaches 0 for $\kappa\rightarrow0$, the magnetization approaches properly the linear-diffusion limit, $4\pi M\approx -H_0(1-2(t/\tau_0)^{1/2}\cdot(x_0/D))$ for $t<\tau_0$. This is found if the functions Γ(z) in (18) and (16) are approximated by $\Gamma(z)\approx z^z\cdot\exp(-z)(2\pi/z)^{1/2}$ for large arguments.
We argue that the scaling relation between t^*, D and H_0, represented by (16), should be valid in relaxation measurements as well as in AC susceptibility studies (with replacement of t^* by the inverse of the AC field frequency, $1/\omega$, and H_0 by the AC field amplitude). For $\kappa\ll1$, close to linearity, the characteristic relaxation time $t^*\sim D^2$ and t^* is independent of H_0. In the opposite limit (strong flux pinning) we find a frequency-independent response and linear scaling between H_0 and D. The predictions for both limiting cases are in agreement with results observed for superconductors but our detailed results for intermediate values of κ are to be verified experimentally.

5 Summary and Conclusion

It has been shown that a pinning potential depending logarithmicaly on current density leads to the same nonlinear diffusion equation for the spatiotemporal evolution of the flux density as a power-law current-voltage characteristic. The exact solution of this equation is derived for the case when the external field, parallel to the surface of a large plate, is applied abruptly. Limiting cases of the solution are identified: either linear processes relevant to the reversible part of the H-T diagram of high-T_c materials or a true critical-state behaviour. The established scaling relation (16) between the characteristic relaxation time, magnetic field and a sample size should be useful in particular for the analysis of measurements performed in the neighbourhood of the irreversibility line and in other cases when strong nonlinearities characterize the electromagnetic response of the superconductor.

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